

C*-ALGEBRA APPROACH TO THE INDEX THEORY OF BOUNDARY VALUE PROBLEMS

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ABSTRACT. Boutet de Monvel's calculus provides a pseudodifferential framework which encompasses the classical differential boundary value problems. In an extension of the concept of Lopatinski and Shapiro, it associates to each operator two symbols: a pseudodifferential principal symbol, which is a bundle homomorphism, and an operator-valued boundary symbol. Ellipticity requires the invertibility of both. If the underlying manifold is compact, elliptic elements define Fredholm operators. Boutet de Monvel [5] showed how then the index can be computed in topological terms. The crucial observation is that elliptic operators can be mapped to compactly supported K -theory classes on the cotangent bundle over the interior of the manifold. The Atiyah-Singer topological index map, applied to this class, then furnishes the index of the operator. Based on this result, Fedosov, Rempel-Schulze and Grubb have given index formulas in terms of the symbols. In this paper we survey how C^* -algebra K -theory, as initiated in [16], can be used to give a proof of Boutet de Monvel's index theorem for boundary value problems, a task carried out in [17], and how the same techniques yield an index theorem for families of Boutet de Monvel operators, detailed in [18]. The key ingredient of our approach is a precise description of the K -theory of the kernel and of the image of the boundary symbol.

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1. BOUTET DE MONVEL'S CALCULUS

Let X be a compact n -dimensional manifold with boundary ∂X , embedded in a closed manifold \tilde{X} of the same dimension. By X° we denote the interior of X . We assume that X is connected and ∂X is nonempty. Given a pseudodifferential operator P on \tilde{X} , we define the truncated pseudodifferential operator $P_+: C^\infty(X) \rightarrow C^\infty(X^\circ)$ as the composition $r^+ P e^+$, where e^+ is extension by zero from X to \tilde{X} and r^+ is the restriction to X° . In general, the functions in the range of P_+ will not be smooth up to the boundary. One therefore assumes that P satisfies the *transmission condition*, a condition on the symbol of P which we recall in (5) and which ensures that both P_+ and $(P^*)_+$, the truncated operator formed from the formal adjoint of P , map smooth functions on X to smooth functions on X .

An operator in Boutet de Monvel's calculus is a matrix

$$(1) \quad A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{array}{c} C^\infty(X, E_1) \\ \oplus \\ C^\infty(\partial X, F_1) \end{array} \rightarrow \begin{array}{c} C^\infty(X, E_2) \\ \oplus \\ C^\infty(\partial X, F_2) \end{array}$$

acting on sections of vector bundles E_1, E_2 over X and F_1, F_2 over ∂X . Here, P is a pseudodifferential operator satisfying the transmission condition; G is a singular

Green operator, T is a trace operator, K is a potential (or Poisson) operator, and S is a pseudodifferential operator on ∂X . All these operators are assumed to be classical; i.e. their symbols have polyhomogeneous expansions in the respective classes. The calculus contains the classical boundary value problems, where P is a differential operator, $G = 0$, and T a differential trace operator. Here $F_1 = 0$; the operators K and S do not appear. It also contains their inverses, provided they exist. In this case, $F_2 = 0$, the operators T and S do not show up, and the inverse to $\begin{pmatrix} P_+ \\ T \end{pmatrix}$ is of the form $(Q_+ + G \ K)$, where K solves the semi-homogeneous problem $Pu = 0, Tu = g$ for given g , and $Q_+ + G$ solves the semi-homogeneous problem $Pu = f, Tu = 0$ for given f . Here Q is a parametrix to P , and G is the correction needed to fulfill the boundary condition. For details, we refer to the monographs by Rempel and Schulze [19] or Grubb [10] as well as to the short introduction [21].

The operators G , K , and T are regularizing in the interior of X . In a collar neighborhood of the boundary, they can be viewed as operator-valued pseudodifferential operators along the boundary. In particular, they have an order assigned to them. The singular Green and the trace operators also have a *class* (or *type*) $d \in \mathbb{N}_0$, related to the order of the derivatives appearing in the boundary condition. The composition of two operators of the form (1) is defined whenever the vector bundles serving as the range of the first operator form the domain of the second. The composition AA' of an operator A' of order m' and class d' with an operator A of order m and class d results in an operator of order $m + m'$ and class $\leq \max(m' + d, d')$. In particular, the composition of two operators of order and class zero is again of order and class zero.

For $E_1 = E_2 = E$ and $F_1 = F_2 = F$, the operators of order and class zero thus form an algebra \mathcal{A}° . Moreover, they extend to bounded operators on the Hilbert space $\mathcal{H} = L^2(X, E) \oplus L^2(\partial X, F)$. In fact, \mathcal{A}° is a $*$ -subalgebra of the algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators on \mathcal{H} , closed under holomorphic functional calculus, cf. [20].¹

Standard reductions –recalled in [17, Section 1.1]– allow to reduce an arbitrary index problem in the calculus (defined by an elliptic Boutet de Monvel operator of arbitrary order and class and acting between different bundles) to the case where the order and class are zero and $E_1 = E_2 = E$ and $F_1 = F_2 = F$. In other words, it suffices to study the index problem for elliptic operators in \mathcal{A}° and we are then free to apply operator-algebraic methods. There is also no loss of generality in the assumption that the manifold X is connected.

We consider an operator A as in (1), with $E_1 = E_2 = E$, $F_1 = F_2 = F$. The pseudodifferential principal symbol $\sigma(A)$ of A is defined to be the principal symbol of the pseudodifferential part P (a smooth bundle morphism), restricted to S^*X . This makes sense as G is regularizing in the interior. The choice of a hermitian structure on E (already needed to define the inner-product of \mathcal{H}) turns the map

$$\mathcal{A}^\circ \ni A \mapsto \sigma(A) \in \text{HOM}(\pi^*E)$$

into a homomorphism of $*$ -algebras. We have denoted by $\pi: S^*X \rightarrow X$ the canonical projection of the co-sphere bundle of X .

¹We use here the definition of order and class in [19] and [21] which differs slightly from that in [10]. It allows us to use the L^2 -space over the boundary instead of the Sobolev space $H^{-1/2}(\partial X, F)$ and gives us better homogeneity properties of the boundary symbols. As the kernel and the cokernel of an elliptic operator in \mathcal{A}° consist of smooth functions, the choice is irrelevant for index theory.

The boundary principal symbol of an $A \in \mathcal{A}_0$ is a smooth endomorphism of

$$(2) \quad (L^2(\mathbb{R}_{\geq 0}) \otimes \pi_\partial^* E|_{\partial X}) \oplus \pi_\partial^* F,$$

with $\pi_\partial: S^* \partial X \rightarrow \partial X$ denoting the canonical projection of the co-sphere bundle of ∂X . It is best described for a trivial one-dimensional bundle and in local coordinates (x', x_n, ξ', ξ_n) for $T^* X$ in a neighborhood of the boundary. Here, G acts like a pseudodifferential operator along the boundary, with an operator-valued symbol taking values in regularizing operators in the normal direction. One way to write this operator-valued symbol is via a so-called symbol kernel $\tilde{g} = \tilde{g}(x', \xi', x_n, y_n)$. For fixed (x', ξ') , this is a rapidly decreasing function in x_n and y_n which acts as an integral operator on $L^2(\mathbb{R}_{\geq 0})$. It satisfies special estimates, combining the usual pseudodifferential estimates in x' and ξ' with those for rapidly decreasing functions in x_n and y_n . The singular Green symbol g of G is defined from the symbol kernel via Fourier and inverse Fourier transform:

$$g(x', \xi', \xi_n, \eta_n) = F_{x_n \rightarrow \xi_n} \overline{F}_{y_n \rightarrow \eta_n} \tilde{g}(x', \xi', x_n, y_n).$$

It has an expansion into homogeneous terms; the leading one we call g_0 . Inverting the operation above, we associate with g_0 a symbol kernel $\tilde{g}_0(x', \xi', x_n, y_n)$ which is rapidly decreasing in x_n and y_n for fixed (x', ξ') . We denote by $g_0(x', \xi', D_n)$ the (compact) operator induced on $L^2(\mathbb{R}_{\geq 0})$ by this kernel. Similarly, K and T have symbol-kernels $\tilde{k}(x', \xi', x_n)$ and $\tilde{t}(x', \xi', y_n)$; these are rapidly decreasing functions for fixed (x', ξ') . The symbols k and t are defined as their Fourier and inverse Fourier transforms. They have asymptotic expansions with leading terms k_0 and t_0 . Via the symbol-kernels \tilde{k}_0 and \tilde{t}_0 one defines $k_0(x', \xi', D_n): \mathbb{C} \rightarrow L^2(\mathbb{R}_{\geq 0})$ as multiplication by $\tilde{k}_0(x', \xi', \cdot)$, while $t_0(x', \xi', D_n): L^2(\mathbb{R}_{\geq 0}) \rightarrow \mathbb{C}$ is the operator $\varphi \mapsto \int \tilde{t}_0(x', \xi', y_n) \varphi(y_n) dy_n$.

We denote by p_0 and s_0 the principal symbols of P and S , respectively. The boundary symbol $\gamma(A)$ of A at (x', ξ') is then defined by

$$(3) \quad \gamma(A)(x', \xi') = \begin{pmatrix} p_0(x', 0, \xi', D_n)_+ + g_0(x', \xi', D_n) & k_0(x', \xi', D_n) \\ t_0(x', \xi', D_n) & s_0(x', \xi') \end{pmatrix}.$$

This gives an invariantly defined operator-valued function on $T^* \partial X$ only up to a choice of a normal coordinate; i.e., we need to restrict ourselves to an atlas whose changes of coordinates, near the boundary, preserve not only the boundary points $\{x_n = 0\}$ but the variable x_n as well [10, Theorem 2.4.11]. The boundary symbol can be viewed as a function on $S^* \partial X$ due to its *twisted homogeneity*,

$$(4) \quad \begin{pmatrix} \kappa_{\lambda^{-1}} & 0 \\ 0 & \text{id} \end{pmatrix} \gamma(A)(x', \lambda \xi') \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & \text{id} \end{pmatrix} = \gamma(A)(x', \xi'), \quad \lambda > 0,$$

with the $L^2(\mathbb{R}_{\geq 0})$ -unitary κ_λ given by $\kappa_\lambda f(t) = \sqrt{\lambda} f(\lambda t)$.

A connection between Toeplitz operators and pseudodifferential operators satisfying the transmission condition turns out to be an essential point for both the computation of the K -theory of the range of the principal boundary symbol and for the proof of the estimate (9) needed to describe its kernel. Let $p \sim \sum p_j$ be the asymptotic expansion of the local symbol p of P into terms $p_j(x, \xi)$, which are positively homogeneous of degree j in ξ for $|\xi| \geq 1$. The transmission condition requires that, for $x_n = 0$ and $\xi = (0, \pm 1)$,

$$(5) \quad D_x^\beta D_\xi^\alpha p_j(x', 0, 0, 1) = (-1)^{j-|\alpha|} D_x^\beta D_\xi^\alpha p_j(x', 0, 0, -1).$$

Hence the limits of $p_0(x', 0, \xi', \xi_n)$ as $\xi_n \rightarrow \pm\infty$ coincide for fixed (x', ξ') , and the function

$$p_{(x', \xi')}(z) = p_0\left(x', 0, \xi', \frac{iz - i}{z + 1}\right), \quad z \in S^1, z \neq -1,$$

extends continuously to S^1 .

We next observe that the image of the Hardy space $H^2(S^1)$ under the unitary map $U: L^2(S^1) \rightarrow L^2(\mathbb{R})$,

$$Ug(t) = \frac{\sqrt{2}}{1+it} g\left(\frac{1-it}{1+it}\right),$$

is equal to $F(L^2(\mathbb{R}_{\geq 0}))$, where F denotes the Fourier transform. The truncated Fourier multiplier

$$(6) \quad \begin{aligned} p_0(x', 0, \xi', D_n)_+ : L^2(\mathbb{R}_{\geq 0}) &\longrightarrow L^2(\mathbb{R}_{\geq 0}) \\ u &\longmapsto F^{-1}(p_0(x', 0, \xi', \cdot)Fu)|_{\mathbb{R}_{\geq 0}} \end{aligned}$$

is therefore equal to $F^{-1}UT_{p_{x', \xi'}}U^{-1}F$, where $T_{p_{x', \xi'}}$ denotes the Toeplitz operator of symbol $p_{x', \xi'}$. It then follows from classical results about Toeplitz operators [6] that

$$(7) \quad \|p_0(x', 0, \xi', D_n)_+\| = \sup_{\xi_n} |p_0(x', 0, \xi', \xi_n)| = \inf_K \|p_0(x', 0, \xi', D_n)_+ + K\|,$$

with the last infimum being taken over all compact operators K on $L^2(\mathbb{R}_{\geq 0})$. In particular, $p_0(x', 0, \xi', D_n)_+$ is compact if and only if $p_0(x', 0, \xi', \xi_n) = 0$ for all $\xi_n \in \mathbb{R}$.

Gohberg [9] and Seeley [22] established the equality between the norm, modulo compacts, of a singular integral operator on a compact manifold and the supremum norm of its symbol. Proofs of that estimate in the language of pseudodifferential appeared in [13, 15]. The following generalization holds for Boutet de Monvel operators (a proof for this result can be found in Rempel and Schulze's book [19, 2.3.4.4]; they credit Grubb and Geymonat [11] for earlier work):

$$(8) \quad \inf_{C \in \mathcal{K}} \|A + C\| = \max\{\|\sigma(A)\|, \|\gamma(A)\|\}, \quad \text{for all } A \in \mathcal{A}^\circ,$$

with \mathcal{K} denoting the ideal of the compact operators on \mathcal{H} , $\|\sigma(A)\|$ the supremum norm of $\sigma(A)$ on S^*X , and $\|\gamma(A)\|$ the supremum over all (x', ξ') in $S^*\partial X$ of $\|\gamma(A)(x', \xi')\|$.

Definition 1. We denote by \mathcal{A} the norm closure in $\mathcal{L}(\mathcal{H})$ of the algebra \mathcal{A}° of all classical Boutet de Monvel operators of order and class zero.

Equation (8) implies, in particular, that σ and γ extend to C^* -algebra homomorphisms defined on \mathcal{A} and taking values in continuous endomorphisms of the bundles π^*E and $(L^2(\mathbb{R}_{\geq 0}) \otimes \pi_\partial^*E|_{\partial X}) \oplus \pi_\partial^*F$, respectively. It also implies that $\ker \gamma \cap \ker \sigma = \mathcal{K}$ and that the quotient \mathcal{A}/\mathcal{K} is isomorphic to the image of the pair (σ, γ) and, in particular, that $A \in \mathcal{A}$ is Fredholm if and only if both $\sigma(A)$ and $\gamma(A)$ are invertible. This description of \mathcal{A}/\mathcal{K} , however, is not explicit enough for K -theory computations.

In the rest of the paper, we will assume that $E = X \times \mathbb{C}$ and $F = \partial X \times \mathbb{C}$. For the general case, all the results can be reformulated in a straightforward way and their proofs can be adapted. A key ingredient in this adaption are canonical Morita equivalences between the algebras acting on functions, the algebras acting

on vectors of functions and the algebras acting on sections of general bundles, giving rise to canonical K-theory isomorphisms. They are based on the well known [16, Subsection 1.5] Morita equivalence between sections of the endomorphism bundle $\text{END}(E)$ of a bundle E and the algebra of functions itself, given by the bimodule of sections of E .

2. THE BOUNDARY-SYMBOL EXACT SEQUENCE

Our description of the kernel of the boundary symbol (or rather of its quotient by the compacts) depends on an estimate for the norm, modulo an ideal of operators in \mathcal{A}° whose closure is larger than the compacts.

Theorem 2. *The principal symbol σ induces a C^* -algebra isomorphism*

$$\ker \gamma/\mathcal{K} \ni [A] \longmapsto \sigma(A) \in C_0(S^*X^\circ),$$

where $C_0(S^*X^\circ)$ denotes the algebra of continuous functions on S^*X which vanish at the boundary.

Sketch of proof. If the upper left corner of the matrix in the right hand side of (3), for an $A \in \mathcal{A}^\circ$, vanishes, then $p_0(x', 0, \xi', D_n)_+$ is compact, since $g_0(x', \xi', D_n)$ is compact. It then follows from (7) that $p_0(x', 0, \xi', \xi_n) = 0$ for all $\xi_n \in \mathbb{R}$ and hence $g_0(x', \xi', D_n) = 0$. This shows that the kernel of γ restricted to \mathcal{A}° is equal to the set \mathcal{I}° of all A as in (1) such that $\sigma(A)$ vanishes at the boundary and, moreover, G , K , T and S are of lower order. To prove that the kernel of γ (defined on the whole algebra \mathcal{A}) is equal to the closure of \mathcal{I}° , which we denote by \mathcal{I} , one needs to use that there exists $C > 0$ such that

$$(9) \quad \inf\{\|A + A'\|, A' \in \mathcal{I}^\circ\} \leq C\|\gamma(A)\|$$

for all $A \in \mathcal{A}^\circ$. The proof of this estimate [16, Lemma 2] combines the above mentioned classical Gohberg-Seeley estimate with (7).

The closed ideal \mathcal{I} can also be described as the C^* -subalgebra of $\mathcal{L}(\mathcal{H})$ generated by all the operators of the form

$$(10) \quad A = \begin{pmatrix} \varphi P \varphi & K_1 \\ K_2 & K_3 \end{pmatrix} : \begin{array}{c} L^2(X) \\ \oplus \\ L^2(\partial X) \end{array} \rightarrow \begin{array}{c} L^2(X) \\ \oplus \\ L^2(\partial X) \end{array},$$

where P is a zero-order classical pseudodifferential operator, φ is (multiplication by) a smooth function with support contained in X° , and K_1 , K_2 and K_3 are compact operators. It then follows from the Gohberg-Seeley estimate that the principal symbol induces the desired isomorphism. \square .

Given $f \in C(X)$, the operator $m(f)$ defined by

$$L^2(X) \oplus L^2(\partial X) \ni (\phi, \psi) \mapsto m(f)(\phi, \psi) = (f\phi, 0) \in L^2(X) \oplus L^2(\partial X)$$

belongs to \mathcal{A} . Abusing notation a little, let us denote also by $m: C(X) \rightarrow \mathcal{A}/\mathcal{K}$ the C^* -algebra homomorphism that maps f to the class of $m(f)$ in the quotient \mathcal{A}/\mathcal{K} and also by γ the map induced by the boundary symbol on the quotient \mathcal{A}/\mathcal{K} with kernel $\ker \gamma/\mathcal{K}$. Taking into account that the isomorphism of Theorem 2 is induced by the principal symbol and that the principal symbol of the multiplication

by a function is equal to the function itself, we then get the following commutative diagram of C^* -algebra exact sequences

$$(11) \quad \begin{array}{ccccccc} 0 & \longrightarrow & C_0(S^*X^\circ) & \longrightarrow & \mathcal{A}/\mathcal{K} & \xrightarrow{\gamma} & \text{im } \gamma & \longrightarrow 0 \\ & & \uparrow m^\circ & & \uparrow m & & \uparrow b & \\ 0 & \longrightarrow & C_0(X^\circ) & \longrightarrow & C(X) & \xrightarrow{r} & C(\partial X) & \longrightarrow 0 \end{array},$$

where m° denotes composition with the bundle projection, r denotes the restriction map and b denotes the homomorphism that maps a function $g \in C(\partial X)$ to the boundary principal symbol of $m(f)$ for some $f \in C(X)$ such that $g = r(f)$.

Let \mathcal{T} denote the Toeplitz algebra on S^1 . It is well-known that \mathcal{T} contains the compact operators and that, as a C^* -algebra, \mathcal{T} is generated by the operators T_φ for $\varphi \in C(S^1)$. By \mathcal{T}_0 we denote the ideal in \mathcal{T} generated by the operators T_φ with φ vanishing at -1 .

By $\mathcal{W}^{1,1}$ we denote the image of the Toeplitz algebra in $\mathcal{L}(L^2(\mathbb{R}_{\geq 0}))$ under the isomorphism sketched after (6), i.e., $\mathcal{W}^{1,1}$ is the C^* -algebra generated by the truncated Fourier multipliers $\varphi(D)_+$, where $\varphi \in C(\mathbb{R})$ has equal limits at $\pm\infty$. We write $\mathcal{W}_0^{1,1}$ for the corresponding image of \mathcal{T}_0 .

Next, we let \mathcal{W} denote the C^* -subalgebra of $\mathcal{L}(L^2(\mathbb{R}_{\geq 0}) \oplus \mathbb{C})$ consisting of all elements whose upper left corner belongs to $\mathcal{W}^{1,1}$, and by \mathcal{W}_0 the ideal where the upper left corner is in $\mathcal{W}_0^{1,1}$. \mathcal{W} is the algebra of *Wiener-Hopf* operators on $\mathbb{R}_{\geq 0}$. The following observation will play an important role:

Lemma 3. *We have $K_0(\mathcal{W}_0) = 0 = K_1(\mathcal{W}_0)$.*

Proof. Denote, for the moment, by \mathcal{K} and \mathcal{K}_\oplus the compact operators on $L^2(S^1)$ and on $L^2(\mathbb{R}_{\geq 0}) \oplus \mathbb{C}$, respectively. Then $C(S^1) \cong \mathcal{T}/\mathcal{K} \cong \mathcal{W}/\mathcal{K}_\oplus$, and we have a short exact sequence

$$0 \rightarrow \mathcal{K}_\oplus \rightarrow \mathcal{W} \rightarrow C(S^1) \rightarrow 0,$$

where the map $\mathcal{W} \rightarrow C(S^1)$ is induced by $\varphi(D)_+ \mapsto \varphi(\frac{iz-i}{z+1})$. The associated 6-term exact sequence is

$$(12) \quad \begin{array}{ccccccc} \mathbb{Z} & \longrightarrow & K_0(\mathcal{W}) & \longrightarrow & \mathbb{Z} & & \\ \uparrow & & & & \downarrow & & \\ \mathbb{Z} & \longleftarrow & K_1(\mathcal{W}) & \longleftarrow & 0 & & \end{array}$$

As there exists a Toeplitz operator of Fredholm index one, there also exists an operator in \mathcal{W} of index one; hence, the index mapping in (12) is surjective. This gives $K_0(\mathcal{W}) = [I] \cdot \mathbb{Z}$ and $K_1(\mathcal{W}) = 0$. The six-term exact sequence associated to

$$0 \longrightarrow \mathcal{W}_0 \longrightarrow \mathcal{W} \longrightarrow \mathbb{C} \longrightarrow 0$$

then shows that $K_0(\mathcal{W}_0) = K_1(\mathcal{W}_0) = 0$. \square

The rest of this section is devoted to a sketch of the proof of:

Theorem 4. *The injective C^* -algebra homomorphism $b : C(\partial X) \rightarrow \text{Im } \gamma$ induces a K -theory isomorphism.*

It follows from our remarks preceding (7) that the image of γ is contained in $C(S^*\partial X, \mathcal{W})$. Using standard arguments of the Boutet de Monvel calculus, one shows that $C(S^*\partial X, \mathcal{W}_0)$ is contained in the image of γ , see [16, Section 3] for details.

Since the intersection of $\text{im } b$ and $C(S^* \partial X, \mathcal{W}_0)$ is trivial, we have

$$(13) \quad C(S^* \partial X, \mathcal{W}_0) \oplus \text{im } b \subseteq \text{im } \gamma,$$

where \oplus denotes the direct sum of Banach spaces, not of C^* -algebras. To prove that the reverse inclusion also holds, we need to consider the C^* -algebra homomorphism λ of $C(S^* \partial X) \otimes \mathcal{W}$ into itself defined by

$$f \otimes \begin{pmatrix} p(D)_+ & * \\ * & * \end{pmatrix} \mapsto p(\infty) f \otimes \begin{pmatrix} \text{Id} & 0 \\ 0 & 0 \end{pmatrix},$$

where Id denotes the identity operator on $L^2(\mathbb{R}_{\geq 0})$. If $F \in \text{im } \gamma$, then $F - \lambda(F)$ belongs to $C(S^* \partial X, \mathcal{W}_0)$, and hence to $\text{im } \gamma$, and then also $\lambda(F)$ belongs to $\text{im } \gamma$.

Next denote by γ_{11} the upper-left corner of γ and suppose that $f \in C(S^* \partial X)$ is such that $(x', \xi') \mapsto f(x', \xi')\text{Id}$ belongs to the image of γ_{11} . Given $\varepsilon > 0$ there exist a pseudodifferential operator P with principal symbol p_0 and a singular Green operator G with principal symbol g_0 such that for all (x', ξ') in $S^* \partial X$

$$(14) \quad \begin{aligned} & \|p_0(x', 0, \xi', D_n) - g_0(x', \xi', D_n) - f(x', \xi')\text{Id}\| \\ &= \|\gamma_{11}(P_+ + G) - f \otimes \text{Id}\| < \varepsilon \end{aligned}$$

As $g_0(x', \xi', D_n)$ is compact, we conclude from (7) that for all (x', ξ') in $S^* \partial X$

$$\sup_{\xi_n} |p_0(x', 0, \xi', \xi_n) - f(x', \xi')| = \inf_{C \in K} \|p_0(x', 0, \xi', D_n) - C - f(x', \xi')\text{Id}\| < \varepsilon.$$

Letting $\xi_n \rightarrow \infty$, the zero-homogeneity of p_0 implies that the left hand side is $\geq |p_0(x', 0, 0, 1) - f(x', \xi')|$. As ε was arbitrary, f is actually independent of the covariable: $f \in C(\partial X)$. This implies that $\lambda(F)$ belongs to $\text{im } b$. Hence equality holds in (13) and the image of γ fits into the following exact sequence of C^* -algebras

$$(15) \quad 0 \rightarrow C(S^* \partial X, \mathcal{W}_0) \rightarrow \text{im } \gamma \rightarrow C(\partial X) \rightarrow 0.$$

This sequence splits via b . Now Lemma 3 and the Künneth formula show that the K -theory of $C_0(S^* \partial X, \mathcal{W}_0)$ vanishes, and Theorem 4 follows from (15).

3. K-THEORY AND INDEX OF BOUTET DE MONVEL OPERATORS I

We start this section recalling some results concerning the K -theory of C^* -algebras, see [17, Section 2]. Let \mathbb{A} be a C^* -algebra. The cone over \mathbb{A} is the C^* -algebra $C\mathbb{A} := \{\phi: [0, 1] \rightarrow \mathbb{A}; \phi \text{ is continuous and } \phi(1) = 0\}$. Since $C\mathbb{A}$ is a contractible C^* -algebra, its K -theory vanishes. The suspension of \mathbb{A} is given by $S\mathbb{A} := \{\phi \in C\mathbb{A}; \phi(0) = 0\}$. If $f: \mathbb{B} \rightarrow \mathbb{A}$ is a C^* -algebra homomorphism, the mapping cone Cf is defined to be $Cf := \{(b, \phi) \in \mathbb{B} \oplus C\mathbb{A}; f(b) = \phi(0)\}$. The projection q onto \mathbb{B} defines a short exact sequence

$$(16) \quad 0 \longrightarrow S\mathbb{A} \xrightarrow{i} Cf \xrightarrow{q} \mathbb{B} \longrightarrow 0,$$

with i denoting the inclusion $i: S\mathbb{A} \ni \phi \mapsto (0, \phi) \in Cf$.

The assignment of the exact sequence (16) to each C^* -algebra homomorphism $f: \mathbb{B} \rightarrow \mathbb{A}$ defines a functor between the corresponding categories (whose morphisms consist of commutative diagrams of homomorphisms or of exact sequences, respectively). This functor is exact. Another important observation is the following:

Lemma 5. *The connecting maps in the standard cyclic 6-term exact sequence associated to (16) are equal, modulo the canonical isomorphisms $K_i(S\mathbb{A}) \cong K_{1-i}(\mathbb{A})$, to the group homomorphisms induced by f .*

If f is additionally surjective, then the map $j: \ker f \rightarrow Cf$, given by $x \mapsto (x, 0)$, induces a K -theory isomorphism, which fits into the commutative diagram

$$\begin{array}{ccccccc} \longrightarrow & K_{i+1}(B) & \longrightarrow & K_i(SA) & \xrightarrow{\hat{\iota}_*} & K_i(Cf) & \xrightarrow{q_*} K_i(B) \longrightarrow \\ & \uparrow = & & \uparrow \delta_{i+1} & & \uparrow j_* & \\ \longrightarrow & K_{i+1}(B) & \xrightarrow{f_*} & K_{i+1}(A) & \longrightarrow & K_i(\ker f) & \longrightarrow K_i(B) \longrightarrow \end{array}$$

where the upper row is the cyclic exact sequence induced by (16), and the lower one is that induced by $0 \rightarrow \ker f \rightarrow B \xrightarrow{f} A \rightarrow 0$.

Applying the short exact sequence (16) to the commutative diagram (11), one obtains the commutative grid:

$$(17) \quad \begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 & \longrightarrow & C_0(X^\circ) & \longrightarrow & C(X) & \xrightarrow{r} & C(\partial X) \longrightarrow 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 & \longrightarrow & Cm^\circ & \longrightarrow & Cm & \longrightarrow & Cb \longrightarrow 0. \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 & \longrightarrow & S(\mathcal{I}/\mathcal{K}) & \longrightarrow & S(\mathcal{A}/\mathcal{K}) & \xrightarrow{S\pi} & S(\mathcal{A}/\mathcal{I}) \longrightarrow 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 & & 0 & & 0 & & 0 \end{array}$$

Next consider the associated long exact sequences in K -theory. By Theorem 4, b induces an isomorphism in K -theory. From Lemma 5 and the cyclic exact sequence of $0 \rightarrow S(\mathcal{A}/\mathcal{I}) \rightarrow Cb \rightarrow C(\partial X) \rightarrow 0$ we then conclude that $K_*(Cb) = 0$. From this in turn we deduce, using the cyclic exact sequence of $0 \rightarrow Cm^\circ \rightarrow Cm \rightarrow Cb \rightarrow 0$, that $Cm^\circ \rightarrow Cm$ induces an isomorphism in K -theory.

We know from Theorem 2 that $\mathcal{I}/\mathcal{K} = \ker \gamma/\mathcal{K} \cong C_0(S^*X^\circ)$. Together with the canonical isomorphism $K_*(S(\mathcal{I}/\mathcal{K})) \cong K_{1-*}(\mathcal{I}/\mathcal{K})$ the left two vertical exact sequences induce the following commutative diagram in K -theory:

$$(18) \quad \begin{array}{ccc} K_0(C_0(X^\circ)) & \longrightarrow & K_0(C(X)) \\ \downarrow m_*^\circ & & \downarrow m_* \\ K_0(C_0(S^*X^\circ)) & \longrightarrow & K_0(\mathcal{A}/\mathcal{K}) \\ \downarrow & & \downarrow \\ K_1(Cm^\circ) & \xrightarrow{\cong} & K_1(Cm) \\ \downarrow & & \downarrow \\ K_1(C_0(X^\circ)) & \longrightarrow & K_1(C(X)) \\ \downarrow m_*^\circ & & \downarrow m_*^\circ \\ K_1(C_0(S^*X^\circ)) & \xrightarrow{i_*} & K_1(\mathcal{A}/\mathcal{K}) \\ \downarrow \alpha & & \downarrow \beta \\ K_0(Cm^\circ) & \xrightarrow[\phi]{\cong} & K_0(Cm) \\ \downarrow & & \downarrow \\ K_0(C_0(X^\circ)) & \longrightarrow & K_0(C(X)). \end{array}$$

We shall now see how this can be used to derive a K -theoretic proof of Boutet de Monvel's index theorem. A crucial ingredient is the following well-known result:

Lemma 6. *A connected compact manifold with non-empty boundary always has a nowhere vanishing vector field.*

This implies that the co-sphere bundle of X has a continuous section s . Composition with s then defines a left inverse for $m^\circ: C_0(X^\circ) \rightarrow C_0(S^*X^\circ)$. This yields a right inverse s' for the map α in (18), and $s'' = i_* \circ s' \circ \phi^{-1}$ yields a right inverse for β . Hence both long exact sequences in (18) split, and that on the right hand side yields the split short exact sequences

$$(19) \quad 0 \longrightarrow K_i(C(X)) \xrightarrow{m_*} K_i(\mathcal{A}/\mathcal{K}) \xrightarrow{\beta} K_{1-i}(Cm) \longrightarrow 0, \quad i = 0, 1.$$

It is worthwhile noting an immediate consequence of this split exactness:

Corollary 7. *Each element of $K_i(\mathcal{A}/\mathcal{K})$ can be written as the sum of an element in the range of m_* and an element in the range of s'' , hence in the range of i_* .*

In order to determine $K_i(Cm) \cong K_i(Cm^\circ)$ we consider the commutative diagram

$$(20) \quad \begin{array}{ccccccc} 0 & \longrightarrow & C_0(T^*X^\circ) & \longrightarrow & C_0(B^*X^\circ) & \xrightarrow{r} & C_0(S^*X^\circ) \longrightarrow 0 \\ & & \pi^*r_0 \uparrow \sim & & & & \uparrow = \\ & & C_0(B^*X^\circ) & \xrightarrow{\pi^*r_0} & C_0(S^*X^\circ) & & \\ & & r_0 \downarrow \sim & & & & \downarrow = \\ & & C_0(X^\circ) & \xrightarrow{\pi^* = m^\circ} & C_0(S^*X^\circ). & & \end{array}$$

Here, π^* denotes pull back from the base to the total space of the bundle, while r and r_0 denote restriction to the boundary of the ball bundle and the zero section of the ball bundle, respectively; \sim denotes homotopy equivalence of C^* -algebras.

We get induced short exact mapping cone sequences

$$(21) \quad \begin{array}{ccccccc} 0 & \longrightarrow & SC_0(S^*X^\circ) & \longrightarrow & Cr & \longrightarrow & C_0(B^*X^\circ) \longrightarrow 0 \\ & & \uparrow = & & \uparrow (\pi^*r_0)_* & & \sim \uparrow \pi^*r_0 \\ 0 & \longrightarrow & SC_0(S^*X^\circ) & \longrightarrow & C(\pi^*r_0) & \longrightarrow & C_0(B^*X^\circ) \longrightarrow 0 \\ & & \downarrow = & & \sim \downarrow (r_0)_* & & \sim \downarrow r_0 \\ 0 & \longrightarrow & SC_0(S^*X^\circ) & \longrightarrow & Cm^\circ & \longrightarrow & C_0(X^\circ) \longrightarrow 0. \end{array}$$

Applying the 5-lemma to the corresponding cyclic exact K -theory sequences we see that the induced maps between the mapping cones are K -theory isomorphisms.

Finally, since r is surjective and $\ker r = C_0(T^*X^\circ)$, Lemma 5 yields the commutative diagram

$$(22) \quad \begin{array}{ccc} K_0(SC_0(S^*X^\circ)) & \longrightarrow & K_0(Cr) \\ \cong \uparrow & & \cong \uparrow j_* \\ K_1(C_0(S^*X^\circ)) & \xrightarrow{\delta} & K_0(C_0(T^*X^\circ)) \end{array},$$

where the lower horizontal arrow is the index mapping for the first row in (20), and the upper horizontal is induced by the first row in (21).

This furnishes natural isomorphisms

$$K_i(Cm) \cong K_i(Cm^\circ) \cong K_i(C_0(T^*X^\circ)).$$

We next consider the commutative diagram

$$(23) \quad \begin{array}{ccccc} K_1(C(X)) & \xrightarrow{m_*} & K_1(\mathcal{A}/\mathcal{K}) & \xrightarrow{\beta} & K_0(Cm) \\ \uparrow & & \uparrow i_* & & \uparrow \cong \\ K_1(C_0(X^\circ)) & \xrightarrow{m^\circ} & K_1(\mathcal{I}/\mathcal{K}) & \xrightarrow{\alpha} & K_0(Cm^\circ) \\ & & a \uparrow \cong & & \uparrow \cong \\ & & K_0(SC_0(S^*X^\circ)) & \longrightarrow & K_0(C(\pi^*r_0)) \\ & & \downarrow = & & \downarrow \cong \\ & & K_0(SC_0(S^*X^\circ)) & \longrightarrow & K_0(Cr) \\ & & c \uparrow \cong & & \uparrow \cong \\ K_1(C_0(S^*X^\circ)) & \xrightarrow{\delta} & K_0(C_0(T^*X^\circ)) & & \downarrow \text{ind}_t \\ & & & & \downarrow \mathbb{Z} \end{array}$$

where the first two rows are portions of (18). The second, third and fourth rows in (23) are portions of the cyclic sequences associated to (21) (notice that, if we use the isomorphism $\mathcal{I}/\mathcal{K} \cong C_0(S^*X^\circ)$ as an identification, then the first column in (17) is equal to the last row in (21)), while the fourth and fifth rows are just (22). Note that the composed isomorphism $c^{-1}a^{-1}: K_1(\mathcal{I}/\mathcal{K}) \rightarrow K_1(C_0(S^*X^\circ))$ in the left row is exactly the map induced by the interior symbol.

Definition 8. *We define the map $p: K_1(\mathcal{A}/\mathcal{I}) \rightarrow K_0(C_0(T^*X^\circ))$ as the composition of all the maps (reverting arrows of isomorphisms when necessary) in the right column in (23), except ind_t , with the map β from $K_1(\mathcal{A}/\mathcal{K})$ to $K_0(Cm)$ in the first row.*

We then infer from (19):

Theorem 9. *$K_i(\mathcal{A}/\mathcal{K})$ fits into the short exact sequence*

$$(24) \quad 0 \longrightarrow K_i(C(X)) \xrightarrow{m_*} K_i(\mathcal{A}/\mathcal{K}) \xrightarrow{p} K_{1-i}(C_0(T^*X^\circ)) \longrightarrow 0, \quad i = 0, 1.$$

The sequence splits, but not naturally.

For $i = 1$, we thus have a natural map

$$(25) \quad K_1(\mathcal{A}/\mathcal{K}) \xrightarrow{p} K_0(C_0(T^*X^\circ)) \cong K_c(T^*X^\circ),$$

where the last isomorphism is the identification of C^* -algebra K -theory with compactly supported K -theory of topological spaces. We can now state:

Theorem 10. *Let $\chi: K_c(T^*X^\circ) \rightarrow \mathbb{Z}$ be the topological index map defined by Atiyah and Singer. For an elliptic boundary value problem $A \in \mathcal{A}$ we then have*

$$(26) \quad \text{ind } A = \chi \circ p([[A]]_1).$$

Here $[[A]]_1$ is the K_1 -class of the class $[A]$ of A in \mathcal{A}/\mathcal{K} , and we have used the identification of the K -theories mentioned above.

Remark 11. Further analysis shows that this map is precisely the map Boutet de Monvel constructed in [5] using deformations of boundary value problems and topological K -theory. See [17, Section 4] for details.

In order to prove Theorem 10, we note that, by Corollary 7, it is sufficient to treat the two cases where $[A]$ is in the range of m_* or in the range of i_* . The elements in the range of m_* are equivalence classes of invertible multiplication operators. Their Fredholm index therefore is zero. On the other hand, the first row in (23) is exact, thus the range of m_* is mapped to zero. Hence both sides of (26) are zero.

If $[A]$ is in the range of the map i_* induced by the inclusion $i: \mathcal{I}/\mathcal{K} \hookrightarrow \mathcal{A}/\mathcal{K}$, then we may assume that A is of the form (10), and the equality of both sides in (26) essentially follows from the Atiyah-Singer index theorem by considering $\varphi P \varphi$ as a pseudodifferential operator on \widetilde{X} .

This proof breaks down in the case of elliptic families. Then it will no longer be true that the map m° has a left inverse. In the next section we will outline an alternative way of computing the K -theory of \mathcal{A}/\mathcal{K} . This approach will extend to the families case and lead to a proof of an index theorem for families, as explained in Section 5.

4. K-THEORY AND INDEX OF BOUTET DE MONVEL OPERATORS II

Let B denote the subalgebra of $C(S^*X)$ consisting of the functions which do not depend on the co-variable over the boundary, that is, $f \in C(S^*X)$ belongs to B if and only if the restriction of f to $S^*X|_{\partial X}$ is of the form $g \circ \pi$, for some $g \in C(\partial X)$, where $\pi: S^*X \rightarrow X$ is the canonical projection. We will denote by S^*X/\sim the quotient of S^*X by the equivalence relation which identifies all $e, f \in S^*X|_{\partial X}$ such that $\pi(e) = \pi(f)$. The algebra B is then canonically isomorphic to $C(S^*X/\sim)$.

Let \mathcal{A}^\dagger denote the C^* -subalgebra of $\mathcal{L}(\mathcal{H})$ generated by all operators of the form

$$(27) \quad A = \begin{pmatrix} P_+ & K_1 \\ K_2 & K_3 \end{pmatrix}: \begin{matrix} L^2(X) \\ \oplus \\ L^2(\partial X) \end{matrix} \rightarrow \begin{matrix} L^2(X) \\ \oplus \\ L^2(\partial X) \end{matrix},$$

where P is a pseudodifferential operator satisfying the transmission condition with principal symbol belonging to B , and K_1, K_2 and K_3 are compact operators. Comparing with (10) it is then clear that $\mathcal{I} = \ker \gamma$ is contained in \mathcal{A}^\dagger .

Proposition 12. *The restriction of σ to \mathcal{A}^\dagger has kernel equal to \mathcal{K} and image equal to B . In other words, the principal symbol induces an isomorphism $\mathcal{A}^\dagger/\mathcal{K} \cong B$.*

Proof: If the upper left corner of an $A \in \mathcal{A}^\dagger$ is P_+ , where P is a pseudodifferential operator satisfying the transmission condition with principal symbol belonging to B , then $\gamma(A) = \sigma(A)|_{S^*X|_{\partial X}} \otimes \text{Id}$, where Id denotes the identity on $L^2(\mathbb{R}_{\geq 0}) \oplus \mathbb{C}$. Since the set of all such A generate \mathcal{A}^\dagger and γ and σ are homomorphisms, we have $\gamma(A) = \sigma(A)|_{S^*X|_{\partial X}} \otimes \text{Id}$ for all $A \in \mathcal{A}^\dagger$. In particular, the kernel of the restriction of σ to \mathcal{A}^\dagger is contained in $\ker \sigma \cap \ker \gamma = \mathcal{K}$. It is equal to \mathcal{K} because it contains all integral operators with smooth kernel; they are Boutet de Monvel operators of order $-\infty$ and class zero.

If $q \in B \cap C^\infty(S^*X)$, then q is the principal symbol of a pseudodifferential operator satisfying the transmission condition (see [19, Theorem 1 of Section 2.3.3.1], for example). The algebra $B \cap C^\infty(S^*X)$ separates points in S^*X/\sim . Hence $\sigma(\mathcal{A}^\dagger)$

is a dense subalgebra of B , which is a closed subalgebra of $C(S^*X)$. This finishes the proof, since the image of a C^* -algebra homomorphism is always closed. \square

The following proposition can be proven by a diagram chase (see [12, Exercise 38, Section 2.2]).

Proposition 13. *Let there be given a commutative diagram of abelian groups with exact rows,*

$$\begin{array}{ccccccc} \cdots & \rightarrow & A'_i & \xrightarrow{f'_i} & B'_i & \xrightarrow{g'_i} & C'_i & \xrightarrow{h'_i} & A'_{i+1} & \rightarrow & \cdots \\ & & \uparrow a_i & & \uparrow b_i & & \uparrow c_i & & \uparrow a_{i+1} & & \\ \cdots & \rightarrow & A_i & \xrightarrow{f_i} & B_i & \xrightarrow{g_i} & C_i & \xrightarrow{h_i} & A_{i+1} & \rightarrow & \cdots \end{array},$$

where each c_i is an isomorphism. Then the sequence

$$\cdots \longrightarrow A_i \xrightarrow{(a_i, -f_i)} A'_i \oplus B_i \xrightarrow{\langle f'_i, b_i \rangle} B'_i \xrightarrow{h_i c_i^{-1} g'_i} A_{i+1} \longrightarrow \cdots$$

is exact, where $\langle f'_i, b_i \rangle$ is the map defined by $\langle f'_i, b_i \rangle(\alpha, \beta) = f'_i(\alpha) + b_i(\beta)$.

Theorem 14. *Let $\iota: \mathcal{A}^\dagger/\mathcal{K} \rightarrow \mathcal{A}/\mathcal{K}$ denote the canonical inclusion. Then*

$$\iota_*: K_*(\mathcal{A}^\dagger/\mathcal{K}) \rightarrow K_*(\mathcal{A}/\mathcal{K})$$

is an isomorphism.

In view of Proposition 12 this furnishes a description of the K -theory of \mathcal{A}/\mathcal{K} in terms of that of a topological space.

Proof: Applying Proposition 13 to the diagram (18), we get the exact sequence

$$(28) \quad \begin{array}{ccccccc} K_0(C_0(X^\circ)) & \rightarrow & K_0(C(X)) \oplus K_0(C_0(S^*X^\circ)) & \rightarrow & K_0(\mathcal{A}/\mathcal{K}) & & \\ \uparrow & & & & \downarrow & & \\ K_1(\mathcal{A}/\mathcal{K}) & \leftarrow & K_1(C(X)) \oplus K_1(C_0(S^*X^\circ)) & \leftarrow & K_1(C_0(X^\circ)) & & \end{array}.$$

We next consider the following diagram of commutative C^* -algebras

$$(29) \quad \begin{array}{ccc} C_0(X^\circ) & \xrightarrow{m^\circ} & C_0(S^*X^\circ) \\ \downarrow & & \downarrow p_2 \\ C(X) & \xrightarrow{p_1} & B \end{array}.$$

As $C_0(X^\circ)$ is canonically isomorphic to

$$\{(f, g) \in C(X) \oplus C_0(S^*X^\circ); p_1(f) = p_2(g)\},$$

the Mayer-Vietoris exact sequence associated to (29) is the exact sequence

$$(30) \quad \begin{array}{ccccccc} K_0(C_0(X^\circ)) & \rightarrow & K_0(C(X)) \oplus K_0(C_0(S^*X^\circ)) & \rightarrow & K_0(B) & & \\ \uparrow & & & & \downarrow & & \\ K_1(B) & \leftarrow & K_1(C(X)) \oplus K_1(C_0(S^*X^\circ)) & \leftarrow & K_1(C_0(X^\circ)) & & \end{array}.$$

The maps $\iota_*: K_i(B) \cong K_i(\mathcal{A}^\dagger/\mathcal{K}) \rightarrow K_i(\mathcal{A}/\mathcal{K})$, $i = 0, 1$, and the identity on the other K -theory groups furnish morphisms from the cyclic sequence (30) to the cyclic sequence (28). The five lemma then shows that ι_* is an isomorphism. \square

Let $\beta: K_1(C(S^*X/\sim)) \rightarrow K_0(C_0(T^*X^\circ))$ denote the connecting map in the standard cyclic exact sequence associated to

$$(31) \quad 0 \longrightarrow C_0(T^*X^\circ) \longrightarrow C(B^*X/\sim) \longrightarrow C(S^*X/\sim) \longrightarrow 0,$$

where B^*X denotes the bundle of closed co-balls over X (which can be regarded as a compactification of T^*X whose points at infinity form the co-sphere bundle S^*X).

Let $e: C_0(T^*X^\circ) \rightarrow C_0(T^*\tilde{X})$ be the map of extension by zero and denote with $j: K_0(C_0(T^*\tilde{X})) \rightarrow K(T^*\tilde{X})$ the canonical isomorphism between C^* -algebra K -theory and topological K -theory groups.

Theorem 15. *If $A \in \mathcal{A}$ is a Fredholm operator, then*

$$(32) \quad \text{ind } A = \chi \circ j \circ e_* \circ \beta \circ \iota_*^{-1}([A]_1)$$

where ind denotes the Fredholm index, $[A]_1$ denotes the K_1 of the class $[A]$ of A in the quotient \mathcal{A}/\mathcal{K} and $\chi: K(T^*\tilde{X}) \rightarrow \mathbb{Z}$ is Atiyah and Singer's topological index for the closed manifold \tilde{X} .

In Theorem 10 above, we stated that $\text{ind } A = \chi \circ p([A]_1)$, using the identification of C^* -algebra K -theory and topological K -theory. Actually, we could have been more precise, because the Atiyah-Singer topological index map is only defined for the cotangent bundle of a *closed* manifold, and the identification involves the maps e_* and j . Thus that formula should actually read

$$(33) \quad \text{ind } A = \chi \circ j \circ e_* \circ p([A]_1).$$

This is what was in fact shown in [17, Theorem 2].

We can infer Theorem 15 from Equation (33) by showing that

$$(34) \quad p = \beta \circ \iota_*^{-1}.$$

For that, let us consider the following commutative diagram of exact sequences

$$(35) \quad \begin{array}{ccccccc} 0 & \longrightarrow & C_0(T^*X^\circ) & \longrightarrow & C(B^*X/\sim) & \xrightarrow{r} & C(S^*X/\sim) & \longrightarrow 0 \\ & & \uparrow = & & \uparrow i^b & & \uparrow i^s & \\ 0 & \longrightarrow & C_0(T^*X^\circ) & \longrightarrow & C(B^*X^\circ) & \xrightarrow{r} & C(S^*X^\circ) & \longrightarrow 0 \end{array} .$$

Recall that β and δ are the index maps associated to the upper and lower sequence, respectively. The naturality of the index map implies that

$$\delta = \beta \circ i_*^s.$$

Up to the isomorphisms of Theorem 2 and Proposition 12, the map i^s in (35) is equal to the canonical inclusion $\mathcal{I}/\mathcal{K} \rightarrow \mathcal{A}^\dagger/\mathcal{K}$. With $i: \mathcal{I}/\mathcal{K} \hookrightarrow \mathcal{A}/\mathcal{K}$ we have

$$p \circ i_* = \delta$$

up to the K -theory isomorphism induced by the C^* -algebra isomorphism of Theorem 2: For that, see the diagram (23) and the remark following it. Since $i = \iota \circ i^s$, this shows that

$$p \circ \iota_* \circ i_*^s = \beta \circ i_*^s;$$

that is, $p \circ \iota_* = \beta$ holds on the image of i_*^s ; or, equivalently, (34) holds on the image of i_* .

In view of Corollary 7, it remains to show that $p \circ \iota_* = \beta$ also holds on the image of m_* . Now, the exactness of (25) implies that $p \circ m_* = 0$. Hence, all that is left to prove Theorem 15 is to show that $\beta \circ m_* = 0$. If an element $x \in K_1(C(S^*X/\sim))$ is represented by an invertible $f \in M_k(C(X))$ (notice that we are using the isomorphism of Proposition 12 as an identification), then x belongs to the image of $r_*: K_1(C(B^*X/\sim)) \rightarrow K_1(C(S^*X/\sim))$ (since f can also be regarded as an invertible of $M_k(C(B^*X/\sim))$). The exactness of the cyclic exact sequence associated to (31) implies that $\beta(x) = 0$.

Theorem 15 is also a particular case (when the space of parameters Y reduces to one point) of Theorem 21, which can be proven independently of Theorem 10.

5. K-THEORY AND INDEX FOR FAMILIES

The index of a continuous function taking values in Fredholm operators acting on a fixed Hilbert space was defined by Jänich [14] and Atiyah [1]. That definition was adapted by Atiyah and Singer [3] to continuous families of elliptic operators acting on the fibers of a fiber bundle whose fibers are closed manifolds. A slight variation of their definitions, for sections of Fredholm operators in a bundle of C^* -algebras, is used in [18] to state and prove Theorem 21 below, which is based on and generalizes Atiyah and Singer's index theorem for families.

Let X , as before, be a compact manifold with boundary, and take $\tilde{X} = 2X$, the double of X . Let Y be a compact Hausdorff space and let $\pi: Z \rightarrow Y$ be a fiber bundle with fiber X and structure group $\text{Diff}(X)$ (equipped with its usual topology). Each $Z_y = \pi^{-1}(y)$ is a compact manifold with boundary, noncanonically diffeomorphic to X . Let $\delta: U \rightarrow \partial X \times [0, 1)$ be a diffeomorphism defined on an open neighborhood of ∂X . The structure group of the bundle π can be reduced [18, Appendix A] to the subgroup G of $\text{Diff}(X)$ consisting of all those ϕ such that $\delta \circ \phi \circ \delta^{-1}: \partial X \times [0, 1/2) \rightarrow \partial X \times [0, 1)$ is of the form $(x', x_n) \mapsto (\varphi(x'), x_n)$ for some diffeomorphism $\varphi: \partial X \rightarrow \partial X$. The elements of G are such that their reflections to the double $2X$ of X are also diffeomorphisms and this allows us to consider the double $2Z$ of Z , a fiber bundle over Y with fiber $2X$ and structure group $\text{Diff}(2X)$. Another consequence of this technicality is that we will then be able to define the boundary principal symbol of a family of Boutet de Monvel (as remarked after (3), the boundary principal symbol is invariantly defined only after we choose a normal coordinate x_n).

We next fix a continuous family of riemannian metrics on Z_y and use them to define the Hilbert spaces $H_y = L^2(Z_y)$. The union $\mathfrak{H} = \bigcup_{y \in Y} H_y$ can be canonically given the structure of a fiber bundle with fiber $H = L^2(X)$ and structure group G ; here G acts on $L^2(X)$ via the representation $\phi \mapsto T_\phi$, $T_\phi(f) = f \circ \phi^{-1}$, which is continuous with respect to the strong operator topology.

To simplify the exposition, we will denote, for the rest of this section, by \mathcal{A} only the upper left corner of what was denoted by \mathcal{A} in Definition 1; i.e., \mathcal{A} is the norm closure in $\mathcal{L}(L^2(X))$ of the algebra of all operators $P_+ + G$, where P is a classical pseudodifferential operator of order zero satisfying the transmission condition and G is a polyhomogeneous singular Green operator of order and class zero. Analogously, for each $y \in Y$, we define \mathcal{A}_y as the norm closure in the bounded operators on $L^2(Z_y)$ of (the upper left corner of) the algebra of all Boutet de Monvel operators on Z_y .

It is well known that the Boutet de Monvel algebra is invariant under diffeomorphisms. Arguing similarly as in [3, Proposition 1.3], one can show, furthermore, that the map

$$(36) \quad G \times \mathcal{A} \ni (\phi, A) \mapsto T_\phi A T_\phi^{-1} \in \mathcal{A}$$

is jointly continuous. This implies that the union $\mathfrak{A} = \bigcup_{y \in Y} \mathcal{A}_y$ can be canonically given the structure of a fiber bundle with fiber \mathcal{A} .

Definition 16. *The continuous sections of the bundle \mathfrak{N} form a C^* -algebra, which we denote by \mathfrak{A} .*

Remark 17. Our approach differs slightly from that of Atiyah and Singer, who work with sections of Fréchet algebras instead of C^* -algebras.

Note that the continuity property (36) is crucial and limits the choice of the algebras in the fibers. It is not possible, for example, to give $\bigcup_y \mathcal{L}(L^2(Z_y))$ in a canonical way the structure a bundle of C^* -algebras (with structure group the unitary group with norm topology).

We recall the construction of the analytic index of families for the present situation.

Similarly as in [3, Proposition (2.2)] and [1, Proposition A5], one can prove:

Proposition 18. *Let \mathfrak{H} and \mathfrak{A} be as above and let $(A_y)_{y \in Y} \in \mathfrak{A}$ be such that, for each y , A_y is a Fredholm operator on H_y . Then there are continuous sections s_1, \dots, s_q of \mathfrak{H} such that the maps*

$$\begin{aligned} \tilde{A}_y : \quad H_y \oplus \mathbb{C}^q &\longrightarrow H_y \oplus \mathbb{C}^q \\ (v, \lambda) &\longmapsto (A_y v + \sum_{j=1}^q \lambda_j s_j(y), 0) \end{aligned}$$

have image equal to $H_y \oplus 0$ for all $y \in Y$ and hence $(\ker \tilde{A}_y)_{y \in Y}$ is a finite-dimensional vector bundle over Y .

Definition 19. *Given $A = (A_y)_{y \in Y} \in \mathfrak{A}$ as in Proposition 18, we denote by $\ker \tilde{A}$ the bundle $(\ker \tilde{A}_y)_{y \in Y}$ and define*

$$\text{ind}_a A = [\ker \tilde{A}] - [Y \times \mathbb{C}^q] \in K(Y).$$

This is independent of the choices of q and of s_1, \dots, s_q and we call it the analytical index of A .

If k is an integer, the algebra $M_k(\mathfrak{A})$ is naturally isomorphic to the algebra of continuous sections of the bundle of algebras $\mathfrak{N}_k = \bigcup_{y \in Y} M_k(\mathcal{A}_y)$, each $M_k(\mathcal{A}_y)$ a

C^* -subalgebra of the bounded operators on H_y^k . We can then define $\text{ind}_a(A)$ if $A = (A_y)_{y \in Y} \in M_k(\mathfrak{A})$ is a section such that each A_y is a Fredholm operator on H_y^k . The projection of such an $A = (A_y)_{y \in Y} \in M_k(\mathfrak{A})$ in $M_k(\mathfrak{A}/\mathfrak{K})$ is invertible and hence defines an element of $K_1(\mathfrak{A}/\mathfrak{K})$. Since $\text{ind}_a(A)$ is invariant under homotopies and perturbations by compact operator valued sections, we get a homomorphism

$$(37) \quad \text{ind}_a : K_1(\mathfrak{A}/\mathfrak{K}) \longrightarrow K(Y).$$

Let us denote by S^*Z the disjoint union of all S^*Z_y . This can canonically be viewed as the total space of a fiber bundle over Y with structure group G . One analogously defines $S^*\partial Z = \bigcup_y S^*\partial Z_y$ and $S^*Z^\circ = \bigcup_y S^*Z_y^\circ$. The families of homomorphisms

$$\sigma_y : \mathcal{A}_y \rightarrow C(S^*Z_y) \quad \text{and} \quad \gamma_y : \mathcal{A}_y \rightarrow C(S^*\partial Z_y, \mathcal{L}(L^2(\mathbb{R}_{\geq 0}))), \quad y \in Y,$$

piece together, yielding C^* -algebra homomorphisms

$$\sigma : \mathfrak{A} \longrightarrow C(S^*Z) \quad \text{and} \quad \gamma : \mathfrak{A} \longrightarrow C(S^*\partial Z, \mathcal{L}(L^2(\mathbb{R}_{\geq 0}))).$$

For this, note in particular that γ_y is well defined because the structure group of the bundle $\pi : Z \rightarrow Y$ leaves the normal coordinate of X invariant, see [10, Theorem 2.4.11].

Similarly as in equation (27) and Proposition 12 we define \mathfrak{A}^\dagger as the C^* -algebra generated by the families of pseudodifferential operators with principal symbol independent of the covariable over boundary points and show that the principal symbol σ induces an isomorphism

$$\bar{\sigma} : \mathfrak{A}^\dagger / \mathfrak{K} \longrightarrow C(S^* Z / \sim),$$

where \mathfrak{K} denotes the continuous compact operator valued sections of \mathfrak{N} and $S^* Z / \sim$ denotes the union of all $S^* Z_y / \sim$, $y \in Y$. The same arguments that prove Theorem 14 also prove that the canonical inclusion $\iota : \mathfrak{A}^\dagger / \mathfrak{K} \rightarrow \mathfrak{A} / \mathfrak{K}$ induces a K -theory isomorphism

$$\iota_* : K_*(\mathfrak{A}^\dagger / \mathfrak{K}) \longrightarrow K_*(\mathfrak{A} / \mathfrak{K}).$$

Analogously as we did around (31), we also denote by $\beta : K_1(C(S^* Z / \sim)) \rightarrow K_0(C_0(T^* Z^\circ))$ the index map in the standard cyclic exact sequence associated to

$$0 \longrightarrow C_0(T^* Z^\circ) \longrightarrow C(B^* Z / \sim) \longrightarrow C(S^* Z / \sim) \longrightarrow 0.$$

Definition 20. *The topological index ind_t is the following composition of maps*

$$\begin{aligned} \text{ind}_t : K_1(\mathfrak{A} / \mathfrak{K}) &\xrightarrow{\bar{\sigma}_* \circ \iota_*^{-1}} K_1(C(S^* Z / \sim)) \xrightarrow{\beta} K_0(C_0(T^* Z^\circ)) \xrightarrow{e_*} K_0(C_0(T^* 2Z)) \\ &\quad \downarrow \text{AS-ind}_t \\ &\quad K(Y), \end{aligned}$$

where $e : C_0(T^* Z^\circ) \rightarrow C_0(T^* 2Z)$ is the map which extends by zero and AS-ind_t denotes the composition of Atiyah and Singer's [3] topological families-index for the bundle of closed manifolds $2Z$ with the canonical isomorphism $K(T^* 2Z) \simeq K_0(C_0(T^* 2Z))$.

We are ready to state the main result of [18]:

Theorem 21. *The two homomorphisms*

$$\text{ind}_a : K_1(\mathfrak{A} / \mathfrak{K}) \rightarrow K(Y) \quad \text{and} \quad \text{ind}_t : K_1(\mathfrak{A} / \mathfrak{K}) \rightarrow K(Y)$$

are equal.

An arbitrary element of $K_1(\mathfrak{A} / \mathfrak{K})$ is of the form $[[A]]_1$, where the inner brackets denote the class modulo compacts of a Fredholm operator valued element A of $M_k(\mathfrak{A})$. Our strategy to prove Theorem 21 in [18] is to derive the equality of the indices from the classical Atiyah-Singer index theorem for families [3, Theorem (3.1)]. To this end we defined a continuous family (in the sense of [3]) of pseudodifferential operators \hat{A} acting on a suitably constructed vector bundle over $2Z$ such that the topological indices of A and of \hat{A} are equal, and similarly the analytical indices of A and \hat{A} are also equal.

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